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Analysis of 90/150 Cellular Automata with Extended Symmetrical Transition Rules

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Abstract.
In this paper we analyze 90/150 cellular automata with extended symmetrical transition rules of various types. The proposed method is an extension of methods for the synthesis of 90/150 CA proposed by Sabater et al. [7] and Cho et al. [10]. Also the method is an extension of the results of Choi et al. [12] for the case of 90/150 CA. By the proposed method, we can compute efficiently characteristic polynomials of large cell CA.

Keywords: Cellular automata, Characteristic polynomial, Symmetrical transition rule, Rule vector.

1. Introduction

Cellular Automata(CA) were originally introduced by Von Neumann in early 1950's in order to study the logical properties of self-reproducing machines [1]. Wolfram in early 1980's suggested a simplified two-state three-neighborhood 1-D CA with cells arranged linearly in one dimension [2]. CA has a simple, regular, modular and cascadable structure with logical neighborhood interconnection. The simple structure of CA with logical interconnections are ideally suited for hardware implementation. For these reasons CA have been used for diverse applications such as pseudorandom-number generation, error-correcting codes, cryptography, and pattern classification, etc. ([5] $\sim$ [10]). Cho et al. ([10],[11]) analyzed characteristic polynomials of group CA and non-group CA. Moreover they proposed efficient methods of synthesis of 90/150 maximum length CA[11]. Sabater et al. [7] and Cho et al. [10] proposed broad classes of cryptographic interleaved sequences generated by linear 90/150 CA obtained by concatenating the basic automaton. In this paper we analyze null-boundary 90/150 CA with

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extended symmetrical transition rules. The proposed method is an extension of methods for the synthesis of CA proposed by Sabater et al. [7] and Cho et al. [10]. Also the method is an extension of the results of Choi et al. [12] for the case of 90/150 CA. By the proposed method, we can compute efficiently characteristic polynomials of large cell CA.

2. Preliminaries

A CA consists of a number of interconnected cells arranged spatially in a regular manner [2]. In most simple case, a CA cell can exhibit two different states(0 or 1) and the next state of each cell depends on the present states of its three neighborhoods including itself. The state s_i^{t+1} of the ith cell at time $(t+1)$ is denoted as

$$s_i^{t+1} = f_i(s_{i-1}^t, s_i^t, s_{i+1}^t),$$

where s_i^t denotes the state of the ith cell at time t and f_i is the next state function called the rule of the CA. If the next state generating logic employs only XOR logic then it is called a *linear rule*. And a CA with all the cells having linear rules is called a *linear CA* [9]. Since a linear CA employs XOR logic only as the next state function, it can be represented as a matrix referred to as the *state transition matrix* over $GF(2)$. An n-cell CA is characterized by an $n \times n$ state transition matrix. The state transition matrix T is constructed as

$$T = \begin{pmatrix} d_1 & a_{1,2} & 0 & \cdots 0 & 0 \\ a_{2,1} & d_2 & a_{2,3} & \cdots 0 & 0 \\ 0 & a_{3,2} & d_3 & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & 0 & \cdots a_{n,n-1} & d_n \end{pmatrix}$$

$$a_{i,j} = \begin{cases} 1 & \text{, if the next state of the } i\text{th cell depends on the present state} \\ & \text{of the } j\text{th cell} \\ 0 & \text{, otherwise.} \end{cases}$$

And $a_{i,i} = d_i$, $i = 1, 2, \cdots, n$.

In this paper, a CA is a null-boundary 90/150 CA fully specified by which cells use 90 and 150. A natural form for the specification of 90/150 CA is an n-tuple $< d_1, d_2, \cdots, d_n >$, called the rule vector, where $d_i = 0$ (resp. 1) if cell i uses rule 90(resp. 150).

A polynomial is said to be a *CA-polynomial* if it is the characteristic polynomial of some 90/150 CA. All irreducible polynomials are CA-polynomials [4]. In [4], Cattell et al. proposed a method for the synthesis of one-dimensional 90/150 Linear Hybrid Group CA(LHGCA) for irreducible polynomial. Cho et al. [11] proposed a new efficient method for the synthesis of one-dimensional 90/150 LHGCA for any CA-polynomial as well as irreducible polynomial by using Lanczos tridiagonalization algorithm. This algorithm is efficient and suitable for all practical applications. Sabater et al. [7] and Cho et al. [10] proposed a method of constructing a linear 90/150 CA with characteristic polynomial $f(x)^2$ by concatenating the basic automaton whose characteristic polynomial is $f(x)$. Successive applications of this result provide one with CA whose characteristic polynomials are $f(x)^2, f(x)^{2^2}, \cdots$.

3. Analysis of CA with extended symmetrical transition rules

In this section, we analyze characteristic polynomials of cellular automata with extended symmetrical transition rules of various types. The characteristic polynomial Δ_n of an n-cell CA $\mathbf{C}_n$ is defined by $\Delta_n = |T_n \oplus xI_n|$ where x is an indeterminate, I_n is the $n \times n$ identity matrix and T_n is the state transition matrix of $\mathbf{C}_n$. For any n-cell 90/150 CA whose state transition matrix is T_n, the minimal polynomial for T_n is the same as the characteristic polynomial for T_n [3]. Let Δ_n be the characteristic polynomial of T_n. Then the following recurrence relation holds:

$$\Delta_n = (x+d_n)\Delta_{n-1}+a_{n-1,n}a_{n,n-1}\Delta_{n-2} \tag{3.1}$$

where $\Delta_1 = x + d_1$, $\Delta_0 = 1$ [4]. (3.1) provides an efficient algorithm to compute Δ_n of a given CA from its rule vector. We denote the characteristic polynomial of *sub-CA* consisting of cells i through j by $\Delta_{i,j}$, where $i \le j$. We simply denote $\Delta_{1,i}$ by Δ_i. Δ_i is said to be a *CA-subpolynomial.* Let T_n^* be the state transition matrix corresponding to the state transition matrix T_n of an n-cell CA as the following

$$T_n^* = \begin{pmatrix} d_n & a_{n,n-1} & 0 & 0 & 0\cdots 0 & 0 & 0 \\ a_{n-1,n} & d_{n-1} & a_{n-1,n-2} & 0 & 0\cdots 0 & 0 & 0 \\ 0 & a_{n-2,n-1} & d_{n-2} & a_{n-2,n-3} & 0\cdots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \cdots a_{2,3} & d_2 & a_{2,1} \\ 0 & 0 & 0 & 0 & 0\cdots 0 & a_{1,2} & d_1 \end{pmatrix}$$

and let Δ_n^* be the characteristic polynomial of T_n^*. Then $\Delta_n^* = \Delta_n$.

Definition 3.1 [12] Let T_n be the state transition matrix of an n-cell CA and let T_n^* be the state transition matrix corresponding to T_n. Let S_{2n} be the following state transition matrix of a $2n$-cell CA $\mathbf{C}_{2n}$.

$$S_{2n} = \left(\begin{array}{cc|ccc} & & 0 & 0 & \cdots 0 \\ & T_n & \vdots & \vdots & \ddots \vdots \\ & & 0 & 0 & \cdots 0 \\ & & a_{n,n+1} & 0 & \cdots 0 \\ 0\cdots 0 & a_{n+1,n} & & & \\ 0\cdots 0 & 0 & & & \\ \vdots \ddots \vdots & \vdots & & T_n^* & \\ 0\cdots 0 & 0 & & & \end{array}\right)$$

Then $\mathbf{C}_{2n}$ is called the *CA with symmetrical transition rules.*

Let $T_n =< a_1, a_2, \cdots, a_n >$ be the state transition matrix of an n-cell 90/150 CA and let $T_n^* =< a_n, a_{n-1}, \cdots, a_1 >$ be the state transition matrix corresponding to T_n. And let $S_{2n} =< a_1, a_2, \cdots, a_n, a_n, \cdots, a_2, a_1 >$ be the state transition matrix of a $2n$-cell 90/150 CA $\mathbf{C}_{2n}$. And let U_{2n} be the characteristic polynomial of S_{2n}. Then the following holds: $U_{2n} = (\Delta_n + \Delta_{n-1})^2$. Let $T_{\overline{n}} =< a_1, \cdots, a_{n-1}, \overline{a_n} >$ be the state transition matrix of an n-cell 90/150 CA and let $\Delta_{\overline{n}}$ be the characteristic polynomial of $T_{\overline{n}}$. Then $\Delta_{\overline{n}} = \Delta_n + \Delta_{n-1}$. Thus $U_{2n} = (\Delta_{\overline{n}})^2$ [12].

Theorem 3.2 [12] Let D_1 be the characteristic polynomial of the 1-cell 90/150 CA with rule vector $< d >$. Let $S_{2n+1} =< a_1, a_2, \cdots, a_n, d, a_n, \cdots, a_2, a_1 >$ be the state transition matrix of a $(2n+1)$-cell 90/150 CA $\mathbf{C}_{2n+1}$. And let U_{2n+1} be the characteristic polynomial of S_{2n+1}. Then

$$U_{2n+1} = D_1 \Delta_n^2.$$

Here we extend the symmetrical transition rules by adding the 2-cell CA $< d_1, d_2 >$ as follows. The recurrence relation between the characteristic polynomial U_{2n+2} of the $(2n+2)$-cell 90/150 CA and Δ_n's is given in the next theorem.

Theorem 3.3 Let $T_n =< a_1, a_2, \cdots, a_n >$ be the state transition matrix of an n-cell 90/150 CA and let D_2 be the characteristic polynomial of the 2-cell 90/150 CA with rule vector $< d_1, d_2 >$. Let $S_{2n+2} =< T_n d_1 d_2 T_n^* >:=< a_1, a_2, \cdots, a_n, d_1, d_2, a_n, \cdots, a_2, a_1 >$ be the state transition matrix of a $(2n+2)$-cell 90/150 CA $\mathbf{C}_{2n+2}$. And let U_{2n+2} be the characteristic polynomial of S_{2n+2}. Then

$$U_{2n+2} = D_2 \Delta_n^2 + (d_1 + d_2)\Delta_n \Delta_{n-1} + \Delta_{n-1}^2$$

Proof. By cofactor expansion along $(n+2)$th row, we have

$$\begin{aligned} U_{2n+2} = &\ \Delta_{n+1}^{r_{d_1}}(x + d_2)\Delta_n + 1 \cdot |A| + 1 \cdot |B| \\ = &\ \Delta_{n+1}^{r_{d_1}}(x + d_2)\Delta_n + 1 \cdot \Delta_n \Delta_n + 1 \cdot \Delta_{n+1}^{r_{d_1}} \Delta_{n-1} \\ = &\ (x + d_2)\{(x + d_1)\Delta_n + \Delta_{n-1}\}\Delta_n + \Delta_n \Delta_n \\ &+ \{(x + d_1)\Delta_n + \Delta_{n-1}\}\Delta_{n-1} \text{ by } (3.1) \\ = &\ D_2 \Delta_n^2 + (d_1 + d_2)\Delta_n \Delta_{n-1} + \Delta_{n-1}^2, \end{aligned}$$

where A is the submatrix obtained by removing the $(n+2)$th row and the $(n+1)$th column of $S_{2n+2} + xI_{2n+2}$ and B is the submatrix obtained by removing the $(n+2)$th row and the $(n+3)$th column of $S_{2n+2} + xI_{2n+2}$, and $\Delta_{n+1}^{r_{d_1}}$ denotes the characteristic polynomial of rule vector $S_{n+1} =< a_1, \cdots, a_n, d_1 >$.

Corollary 3.4 Let $T_n =< a_1, a_2, \cdots, a_n >$ be the state transition matrix of an n-cell 90/150 CA. Let $S_{2n+2} =< T_n d d T_n^* >$ be the state transition matrix of

Table 1. Characteristic polynomials for $(d_1, d_2) = (0, 1)$

(a_1, a_2, a_3)	rule vector for $(d_1, d_2) = (0, 1)$	characteristic polynomial U_8
(0,0,0)	$< 0, 0, 0, \mathbf{0}, \mathbf{1}, 0, 0, 0 >$	$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 1$(irreducible)
(0,0,1)	$< 0, 0, 1, \mathbf{0}, \mathbf{1}, 1, 0, 0 >$	$x^8 + x^7 + x^4 + x^3 + x^2 + x + 1$(irreducible)
(0,1,0)	$< 0, 1, 0, \mathbf{0}, \mathbf{1}, 0, 1, 0 >$	$(x^6 + x^4 + x^3 + x + 1)(x^2 + x + 1)$
(0,1,1)	$< 0, 1, 1, \mathbf{0}, \mathbf{1}, 1, 1, 0 >$	$(x^6 + x^3 + 1)(x^2 + x + 1)$
(1,0,0)	$< 1, 0, 0, \mathbf{0}, \mathbf{1}, 0, 0, 1 >$	$(x^6 + x^4 + x^3 + x + 1)(x^2 + x + 1)$
(1,0,1)	$< 1, 0, 1, \mathbf{0}, \mathbf{1}, 1, 0, 1 >$	$(x^6 + x^3 + 1)(x^2 + x + 1)$
(1,1,0)	$< 1, 1, 0, \mathbf{0}, \mathbf{1}, 0, 1, 1 >$	$x^8 + x^7 + x^6 + x^5 + x^2 + x + 1$(primitive)
(1,1,1)	$< 1, 1, 1, \mathbf{0}, \mathbf{1}, 1, 1, 1 >$	$x^8 + x^7 + x^2 + x + 1$(primitive)

Table 2. Characteristic polynomials for $(d_1, d_2) = (0, 1)$

$T_8 =< a_1, \cdots, a_8 >$	characteristic polynomial U_{18} of $< T_8 01 T_8^* >$
$< 0, 0, 0, \mathbf{0}, \mathbf{1}, 0, 0, 0 >$	$(x + 1)^2(x^{11} + x^{10} + x^3 + x + 1)(x^5 + x^3 + 1)$
$< 0, 0, 1, \mathbf{0}, \mathbf{1}, 1, 0, 0 >$	$(x^{14} + x^{13} + x^{11} + x^9 + x^2 + x + 1)(x^4 + x + 1)$
$< 0, 1, 0, \mathbf{0}, \mathbf{1}, 0, 1, 0 >$	$(x^{12} + x^{10} + x^9 + x^8 + x^6 + x^2 + 1)(x^6 + x^5 + x^4 + x^2 + 1)$
$< 0, 1, 1, \mathbf{0}, \mathbf{1}, 1, 1, 0 >$	$(x^6 + x + 1)(x^{12} + x^{11} + x^8 + x^5 + x^4 + x^2 + 1)$
$< 1, 0, 0, \mathbf{0}, \mathbf{1}, 0, 0, 1 >$	$(x^{14} + x^{13} + x^{11} + x^{10} + x^8 + x^6 + x^5 + x^2 + 1)(x^4 + x + 1)$
$< 1, 0, 1, \mathbf{0}, \mathbf{1}, 1, 0, 1 >$	$x(x + 1)^2(x^6 + x^3 + 1)(x^7 + x^3 + x^2 + x + 1)(x^2 + x + 1)$
$< 1, 1, 0, \mathbf{0}, \mathbf{1}, 0, 1, 1 >$	$(x^2 + x + 1)(x^8 + x^7 + x^6 + x^4 + x^2 + x + 1)$ $(x^8 + x^7 + x^6 + x^5 + x^4 + x^2 + 1)$
$< 1, 1, 1, \mathbf{0}, \mathbf{1}, 1, 1, 1 >$	$(x^{10} + x^8 + x^6 + x^5 + x^2 + x + 1)(x^8 + x^7 + x^6 + x^5 + x^4 + x + 1)$

a $(2n+2)$-cell 90/150 CA $\mathbf{C}_{2n+2}$. And let U_{2n+2} be the characteristic polynomial of S_{2n+2}. Then

$$U_{2n+2} = (E_2 \Delta_n + \Delta_{n-1})^2$$

, where $E_2 = x + \overline{d}$.

For the case $d_1 = d_2 = \overline{a_n}$, the characteristic polynomial U_{2n} of rule vector $< a_1, \cdots, a_{n-1}, \overline{a_n}, \overline{a_n}, a_{n-1}, \cdots, a_1 >$ is as the following:

$$U_{2(n-1)+2} = (E_2 \Delta_{n-1} + \Delta_{n-2})^2 = \Delta_n^2,$$

where $E_2 = x + a_n$.

Remark A Theorem 3.3 is an extension of a method of constructing a 90/150 CA with characteristic polynomial $f(x)^{2^m}$ proposed by Sabater et al. [7] and Cho et al. [10].

Example 3.5 Consider the case $d_1 = 0, d_2 = 1$:
In Theorem 3.3, $U_8 = D_2 \Delta_3^2 + \Delta_3 \Delta_2 + \Delta_2^2$. From $\Delta_3 = (x + a_1)(x + a_2)(x +$

$a_3)+a_1+a_3$ and $\Delta_2 = (x+a_1)(x+a_2)+1$, we obtain Table 1. Using $U_{18} = (x^2+x+1)\Delta_8^2+\Delta_8\Delta_7+\Delta_7^2$, the characteristic polynomials for the case $(d_1, d_2) = (0, 1)$ are in Table 2.

Here, we extend the symmetrical transition rules by adding the 3-cell CA $< d_1, d_2, d_3 >$ as follows. The recurrence relation between the characteristic polynomial U_{2n+3} of the $(2n+3)$-cell 90/150 CA and Δ_n's is given in the next theorem.

Theorem 3.6 Let $T_n =< a_1, a_2, \cdots, a_n >$ be the state transition matrix of an n-cell 90/150 CA and let D_3 be the characteristic polynomial of the 3-cell 90/150 CA with rule vector $< d_1, d_2, d_3 >$. Let $S_{2n+3} =< T_n d_1 d_2 d_3 T_n^* >:=< a_1, a_2, \cdots, a_n, d_1, d_2, d_3, a_n, \cdots, a_2, a_1 >$ be the state transition matrix of a $(2n+3)$-cell 90/150 CA $\mathbf{C}_{2n+3}$. And let U_{2n+3} be the characteristic polynomial of S_{2n+3}. Then

$$U_{2n+3} = D_3\Delta_n^2 + (d_1+d_3)(x+d_2)\Delta_n\Delta_{n-1} + (x+d_2)\Delta_{n-1}^2$$

Proof. By cofactor expansion along $(n+3)$th row, we have

$$\begin{aligned}
U_{2n+3} =\ & \Delta_{n+2}^{r_{d_1,d_2}}(x+d_3)\Delta_n + 1\cdot|A| + 1\cdot|B| \\
=\ & \Delta_{n+2}^{r_{d_1,d_2}}(x+d_3)\Delta_n + \Delta_{n+1}^{r_{d_1}}\Delta_n + \Delta_{n+2}^{r_{d_1,d_2}}\Delta_{n-1} \\
=\ & \{(x+d_2)\Delta_{n+1}^{r_{d_1}} + \Delta_n\}(x+d_3)\Delta_n + \{(x+d_1)\Delta_n + \Delta_{n-1}\}\Delta_n \\
& + \{(x+d_2)\Delta_{n+1}^{r_{d_1}} + \Delta_n\}\Delta_{n-1} \text{ by (3.1)} \\
=\ & \{(x+d_2)\Delta_{n+1}^{r_{d_1}} + \Delta_n\}(x+d_3)\Delta_n + (x+d_1)\Delta_n^2 + (x+d_2)\Delta_{n+1}^{r_{d_1}}\Delta_{n-1} \\
=\ & \{(x+d_2)[(x+d_1)\Delta_n + \Delta_{n-1}] + \Delta_n\}(x+d_3)\Delta_n + (x+d_1)\Delta_n^2 \\
& + (x+d_2)\{(x+d_1)\Delta_n + \Delta_{n-1}\}\Delta_{n-1} \text{ by (3.1)} \\
=\ & \{(x+d_1)(x+d_2)(x+d_3) + d_1 + d_3\}\Delta_n^2 + (d_1+d_3)(x+d_2)\Delta_n\Delta_{n-1} \\
& + (x+d_2)\Delta_{n-1}^2 \\
=\ & D_3\Delta_n^2 + (d_1+d_3)(x+d_2)\Delta_n\Delta_{n-1} + (x+d_2)\Delta_{n-1}^2,
\end{aligned}$$

where A is the submatrix obtained by removing the $(n+3)$th row and the $(n+2)$th column of $S_{2n+3} + xI_{2n+3}$ and B is the submatrix obtained by removing the $(n+3)$th row and the $(n+4)$th column of $S_{2n+3} + xI_{2n+3}$, and $\Delta_{n+1}^{r_{d_1,d_2}}$ denotes the characteristic polynomial of rule vector $S_{n+2} =< a_1, \cdots, a_n, d_1, d_2 >$.

Remark B In the case $d_1 = d_3$, Theorem 3.6 is an extended result of Theorem 3.2.

Using $U_{23} = (x^3+x^2+1)\Delta_{10}^2 + x\Delta_{10}\Delta_9 + x\Delta_9^2$, the characteristic polynomials for the case $(d_1, d_2, d_3) = (0, 0, 1)$ are in Table 3. Here, we extend the symmetrical transition rules by adding the 4-cell CA $< d_1, d_2, d_3, d_4 >$ as follows. The recurrence relation between the characteristic polynomial U_{2n+4} of the $(2n+4)$-cell 90/150 CA and Δ_n's is given in the next theorem.

Theorem 3.7 Let $T_n =< a_1, a_2, \cdots, a_n >$ be the state transition matrix of an n-cell 90/150 CA. And let D_4 be the characteristic polynomial of the 4-cell

90/150 CA with rule vector $< d_1, d_2, d_3, d_4 >$. Let $S_{2n+4} =< T_n d_1 d_2 d_3 d_4 T_n^* >:=< a_1, a_2, \cdots, a_n, d_1, d_2, d_3, d_4, a_n, \cdots, a_2, a_1 >$ be the state transition matrix of a $(2n+4)$-cell 90/150 CA $\mathbf{C}_{2n+4}$. And let U_{2n+4} be the characteristic polynomial of S_{2n+4}. Then

$$\begin{aligned} U_{2n+4} = \; & D_4 \Delta_n^2 + \{(d_1+d_4)(x+d_2)(x+d_3) + (d_1+d_2+d_3+d_4)\}\Delta_n \Delta_{n-1} \\ & + \{(x+d_2)(x+d_3)+1\}\Delta_{n-1}^2. \end{aligned}$$

Proof. By cofactor expansion along $(n+4)$th row, we have

$$\begin{aligned} U_{2n+4} = \; & \Delta_{n+3}^{r_{d_1,d_2,d_3}}(x+d_4)\Delta_n + 1\cdot|A| + 1\cdot|B| \\ = \; & \Delta_{n+3}^{r_{d_1,d_2,d_3}}(x+d_4)\Delta_n + \Delta_{n+2}^{r_{d_1,d_2}}\Delta_n + \Delta_{n+3}^{r_{d_1,d_2,d_3}}\Delta_{n-1} \\ = \; & \{(x+d_3)\Delta_{n+2}^{r_{d_1,d_2}} + \Delta_{n+1}^{r_{d_1}}\}(x+d_4)\Delta_n + \Delta_{n+2}^{r_{d_1,d_2}}\Delta_n \\ & + \{(x+d_3)\Delta_{n+2}^{r_{d_1,d_2}} + \Delta_{n+1}^{r_{d_1}}\}\Delta_{n-1} \text{ by (3.1)} \\ = \; & \{(x+d_3)(x+d_4)\Delta_n + \Delta_n + (x+d_3)\Delta_{n-1}\}\Delta_{n+2}^{r_{d_1,d_2}} \\ & + \{(x+d_4)\Delta_n + \Delta_{n-1}\}\Delta_{n+1}^{r_{d_1}} \\ = \; & \{(x+d_3)[(x+d_4)\Delta_n + \Delta_n + (x+d_3)\Delta_{n-1}\}\{(x+d_2)\Delta_{n+1}^{r_{d_1}} + \Delta_n\} \\ & + \{(x+d_4)\Delta_n + \Delta_{n-1}\}\Delta_{n+1}^{r_{d_1}} \text{ by (3.1)} \\ = \; & \{(x+d_2)(x+d_3)(x+d_4)\Delta_n + (x+d_2)\Delta_n + (x+d_4)\Delta_n \\ & + (x+d_2)(x+d_3)\Delta_{n-1} + \Delta_{n-1}\}\Delta_{n+1}^{r_{d_1}} + \{(x+d_3)(x+d_4)+1\}\Delta_n^2 \\ & + (x+d_3)\Delta_n\Delta_{n-1} \\ = \; & \{(x+d_2)(x+d_3)(x+d_4)\Delta_n + (x+d_2)\Delta_n + (x+d_4)\Delta_n \\ & + (x+d_2)(x+d_3)\Delta_{n-1} + \Delta_{n-1}\}\{(x+d_1)\Delta_n + \Delta_{n-1}\} \\ & + \{(x+d_3)(x+d_4)+1\}\Delta_n^2 + (x+d_3)\Delta_n\Delta_{n-1} \text{ by (3.1)} \\ = \; & D_4\Delta_n^2 + \{(d_1+d_4)(x+d_2)(x+d_3) + (d_1+d_2+d_3+d_4)\}\Delta_n\Delta_{n-1} \\ & + \{(x+d_2)(x+d_3)+1\}\Delta_{n-1}^2, \end{aligned}$$

where A is the submatrix obtained by removing the $(n+4)$th row and the $(n+3)$th column of $S_{2n+4} + xI_{2n+4}$ and B is the submatrix obtained by removing the $(n+4)$th row and the $(n+5)$th column of $S_{2n+4} + xI_{2n+4}$, and $\Delta_{n+1}^{r_{d_1,d_2,d_3}}$ denotes the characteristic polynomial of rule vector $S_{n+3} =< a_1, \cdots, a_n, d_1, d_2, d_3 >$.

Example 3.8 Let $d_1 = d_4$ and $d_2 = d_3$. And let E_4 be the characteristic polynomial of $< d_1, \overline{d_2} >$. Then

$$\begin{aligned} U_{2n+4} &= D_4\Delta_n^2 + \{(x+d_2)(x+d_2)+1\}\Delta_{n-1}^2 \\ &= (E_4\Delta_n + (x+\overline{d_2})\Delta_{n-1})^2 \end{aligned}$$

Using $U_{10} = (x^4+x^3+x^2+1)\Delta_3^2 + (x^2+1)\Delta_3\Delta_2 + (x^2+1)\Delta_2^2$, the characteristic polynomials for the case (d_1, d_2, d_3, d_4) are in Table 4. Table 4 shows the primitive polynomial of degree 10 generated from 3-cell 90/150 CA.

Remark C In the cae $d_1 = d_4$, Theorem 3.7 is an extended result of Theorem 3.3.

Table 3. Characteristic polynomials for $(d_1, d_2, d_3) = (0, 0, 1)$

$T_{10} =< a_1, \cdots, a_{10} >$	characteristic polynomial U_{23} of $< T_{10}001T_{10}^* >$
$< 0,0,0,0,0,0,0,0,0,0 >$	$(x^{16}+x^{15}+x^{14}+x^{10}+x^5+x^2+1)(x^3+x^2+1)$ (x^4+x^3+1)
$< 0,0,0,0,0,0,0,0,0,1 >$	$x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{13}+x^{12}+x^{11}+x^{10}$ $+x^7+x^6+x^5+x^4+x^3+x^2+1$ (primitive)
$< 0,0,0,0,0,0,0,0,1,0 >$	$(x^{13}+x^{11}+x^6+x^5+x^4+x^2+1)(x^{10}+x^9+x^4+x^2+1)$
$< 0,0,0,0,0,0,0,0,1,1 >$	$x(x^4+x^3+1)(x^{18}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9$ $+x^8+x^7+x^5+x^4+x^3+1)$
$< 0,0,0,0,0,0,0,1,0,0 >$	$(x^{17}+x^{15}+x^{14}+x^{12}+x^{10}+x^8+x^5+x^4+x^3+x^2+1)$ $(x^6+x^5+x^3+x^2+1)$
$< 0,0,0,0,0,0,0,1,0,1 >$	$(x^3+x^2+1)(x^9+x^8+x^7+x^6+x^4+x^3+1)$ $(x^{11}+x^{10}+x^8+x^7+x^5+x^3+1)$
$< 0,0,0,0,0,0,0,1,1,0 >$	$x^{23}+x^{22}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}$ $+x^{13}+x^{12}+x^7+x^6+x^5+x^4+1$(primitive)
$< 0,0,0,0,0,0,0,1,1,1 >$	$x(x^{15}+x^{13}+x^8+x^5+1)(x^7+x^6+1)$
$< 0,0,0,0,0,0,1,0,0,0 >$	$(x^{20}+x^{18}+x^{17}+x^{16}+x^{14}+x^{11}+x^9+x^5+x^3+x^2+1)$ (x^3+x^2+1)
$< 0,0,0,0,0,0,1,0,0,1 >$	$x(x^{22}+x^{21}+x^{16}+x^{15}+x^4+x^3+1)$

Table 4. Characteristic polynomials for (d_1, d_2, d_3, d_4)

$T_{10} =< a_1, \cdots, a_{10} >$	characteristic polynomial U_{10} of $< T_3d_1d_2d_3d_4T_3^* >$
$< 1,1,1,\mathbf{0,1,0,1},1,1,1 >$	$x^{10}+x^8+x^6+x^4+x^2+x+1$(primitive)
$< 0,0,0,\mathbf{0,1,0,1},0,0,0 >$	$x^{10}+x^6+x^2+x+1$(irreducible)
$< 1,0,1,\mathbf{0,1,0,0},1,0,1 >$	$(x^6+x+1)(x^4+x^3+x^2+x+1)$
$< 0,1,0,\mathbf{1,1,0,1},0,1,0 >$	$(x^4+x^3+1)(x^6+x^4+x^2+x+1)$
$< 0,0,1,\mathbf{1,0,1,1},1,0,0 >$	$x^{10}+x^9+x^8+x^7+x^2+x+1$(irreducible)
$< 0,1,0,\mathbf{0,0,1,0},0,1,0 >$	$x^{10}+x^9+x^6+x^5+x^4+x^3+x^2+x+1$(primitive)
$< 0,1,0,\mathbf{0,1,1,0},0,1,0 >$	$x^2(x^4+x^3+1)^2$
$< 1,1,0,\mathbf{0,1,1,0},0,1,1 >$	$(x^5+x^3+x^2+x+1)^2$
$< 1,1,1,\mathbf{0,0,0,0},1,1,1 >$	$(x+1)^2(x^4+x^3+x^2+x+1)^2$
$< 1,0,1,\mathbf{0,0,0,1},1,0,1 >$	$(x+1)^2(x^8+x^7+x^2+x+1)$

4. Conclusion

In this paper, we analyzed the properties of 90/150 CA with extended symmetrical transition rules. The proposed method is an extension of methods for the synthesis of 90/150 CA proposed by Sabater et al. [7] and Cho et al.[10]. Also we extended the results of Choi et al. [12] for the case 90/150 CA. By the proposed method, we can compute efficiently characteristic polynomial of large cell 90/150 CA.

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